

A GENERIC GLOBAL TORELLI THEOREM FOR CERTAIN HORIKAWA SURFACES

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ABSTRACT. Algebraic surfaces of general type with $q = 0$, $p_g = 2$ and $K^2 = 1$ have been studied by Horikawa [Hor76]. In this paper we consider a 16-dimensional family of special Horikawa surfaces which are certain bidouble covers of \mathbb{P}^2 . The construction is motivated by that of special Kunev surfaces (cf. [Kyn77] [Cat79] [Cat80] [Tod80]) which are counterexamples for infinitesimal Torelli and generic global Torelli problem. The main result of the paper is a generic global Torelli theorem for special Horikawa surfaces. To prove the theorem, we relate the periods of special Horikawa surfaces to the periods of certain lattice polarized $K3$ surfaces using eigenperiod maps (see [DK07]) and then apply a Torelli type result proved by Laza [Laz09].

INTRODUCTION

There are two particular situations where the period map plays an essential role for studying moduli spaces, namely principally polarized abelian varieties and (lattice) polarized $K3$ surfaces. In these cases, the period domains are Hermitian symmetric domains and the period maps are both injective and dominant. It is an interesting problem to find more examples where the period maps are injective and the images lie in certain Mumford-Tate subdomains which are locally Hermitian symmetric (but the Griffiths infinitesimal period relations may be non-trivial on the ambient period domains). We mention the examples previously studied by Allcock, Carlson, and Toledo [ACT02, ACT11], Kondo [Kon00], Borcea [Bor97], Voisin [Voi93], Rohde [Roh09], Garbagnati and van Geemen ([GvG10]).

The general problem of determining whether the period map is injective is called the Torelli problem. There are four types of Torelli problem (we follow the terminology of [Cat84]): local Torelli (whether the differential of the period map is injective), infinitesimal Torelli (local Torelli for the semi-universal deformation), global Torelli (whether the period map is injective) and generic global Torelli (whether the period map is injective over a open dense subset).

Various Torelli theorems have been proved for a large class of varieties (see for example [Cat84]). However, Kunev [Kyn77] constructed a counterexample for the infinitesimal Torelli and generic global Torelli problem (see also [Cat79] [Cat80] [Tod80]). Let us briefly recall the construction. Let C_1, C_2 be two smooth plane cubic curves intersecting transversely and L be a general line. Let X be the $(\mathbb{Z}/2\mathbb{Z})^2$ -cover of \mathbb{P}^2 branched along $C_1 + C_2 + L$. Then X is a minimal algebraic surface with $q(X) = 0$, $p_g(X) = 1$ and $K_X^2 = 1$. Following [Cat79] such surfaces X are *special Kunev surfaces* whose bicanonical maps are Galois covers of \mathbb{P}^2 . The infinitesimal period map and period map for special Kunev surfaces both have 2-dimensional

fibers (the rough reason is that the period map only sees the Hodge structures of the intermediate $K3$ surfaces obtained as the desingularizations of the double planes along $C_1 + C_2$).

One may ask is it possible to modify the construction or the period map for Kunev surfaces to get Torelli. Usui [Usu83] and Letizia [Let84] considered the complement of the canonical curve $\Lambda \subset X$ and the mixed Hodge structure $H^2(X - \Lambda)$, and proved infinitesimal mixed Torelli and generic global mixed Torelli for special Kunev surfaces X respectively. Our idea is to modify the branch data (see also [Par91a, Def. 2.1]) $C_1 + C_2 + L$. Specifically, we consider the $(\mathbb{Z}/2\mathbb{Z})^2$ -cover S of \mathbb{P}^2 along a smooth quintic C together with two generic lines L_0 and L_1 . The surfaces S are minimal surfaces with $q(S) = 0$, $p_g(S) = 2$ and $K_S^2 = 1$ which have been studied by Horikawa [Hor76]. We shall call such bidouble covers S *special Horikawa surfaces*. We should remark that these surfaces S are also mentioned in the recent preprint [Gar16] by Garbagnati. But her perspective (classify the possible branch loci of a smooth double cover of a $K3$ surface) is quite different from ours.

Let us explain why we want to modify the branch data in this way. On one hand, there are two (lattice polarized) $K3$ surfaces hidden in the construction of a special Horikawa surface S (this is also observed in [Gar16]). Namely, one resolves the singularities of the double cover of \mathbb{P}^2 branched along $C + L_1$ (resp. $C + L_0$) and gets X_0 (resp. X_1) which is a $K3$ surface. On the other hand, it is also natural to study the action of the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ on the periods of the bidouble covers S . We shall show that the eigenperiods (cf. [DK07, §7]) of S are determined by the Hodge structures of the $K3$ surfaces X_0 and X_1 , and apply the global Torelli theorem for (lattice polarized) $K3$ surfaces (a modified version is needed, see the next paragraph) to prove a generic global Torelli theorem for special Horikawa surfaces.

By the work of Pardini [Par91a, Thm. 2.1] the isomorphism classes of the bidouble covers S are determined by the branch data $C + L_0 + L_1$. This can be used to construct the coarse moduli space \mathcal{M} for special Horikawa surfaces (the moduli of special Kunev surfaces has been constructed in a similar manner, see [Usu89]). It also follows that we need the global Torelli theorem [Laz09, Thm. 4.1] for degree 5 pairs (C, L) consisting of plane quintics C and lines L (up to projective equivalence). A key point is that one needs to choose a suitable arithmetic group as explained in [Laz09, Prop. 4.22].

A typical way to prove a generic global Torelli is to study the infinitesimal variation of Hodge structure and first prove a variational Torelli theorem (cf. [CDT87]). Our approach is different. One advantage is that we are able to describe the open dense subset over which the period map is injective explicitly. We suspect that variational Torelli fails for special Horikawa surfaces (otherwise by op. cit. the global Torelli holds for any discrete group of the automorphism group of the period domain, as long as the period map is well-defined, which seems not true; see also [Hay14]).

After “labeling” the lines L_0 and L_1 , we obtain a period map (using the period maps for the degree 5 pairs (C, L_0) and (C, L_1)) from (a double cover of) the moduli \mathcal{M} of special Horikawa surfaces S to a product of two arithmetic quotients of Type IV domains. The period map is generically injective. Therefore, special Horikawa surfaces are along the lines of the examples mentioned at the beginning of the

paper. In an ongoing project with Laza, we use this period map as our guide to compactify the moduli space of special Horikawa surfaces.

A few words on the structure of the paper. The construction of special Horikawa surfaces is given in Section 1. As is well-known (cf. [Hor76]) the canonical model of an algebraic surface with $q = 0$, $p_g = 2$ and $K^2 = 1$ is a degree 10 hypersurface in the weighted projective space $\mathbb{P}(1, 1, 2, 5)$. We shall give the equations for (the canonical models of) special Horikawa surfaces and use Griffiths residue to study the decomposition of the Hodge structures. The infinitesimal Torelli problem will be discussed in Section 2. Usui [Usu78] has proved the infinitesimal Torelli theorem for nonsingular weighted complete intersections satisfying certain conditions which will be checked for special Horikawa surfaces. A second proof will also be included which can be viewed as a boundary case of [Par91b, Thm. 3.1] or [Par98, Thm. 4.2] and might be of independent interest. In Section 3 we discuss the generic global Torelli problem for special Horikawa surfaces and prove the main result.

Acknowledgement. The work is partly motivated by the recent project by Green, Griffiths, Laza and Robles on studying degenerations of “H-surfaces” (which are of general type with $p_g = K^2 = 2$) using Hodge theory. We thank Griffiths for his interest in this paper. The second named author would also like to thank Laza for several useful discussions.

1. SPECIAL HORIKAWA SURFACES

Let C be a smooth plane quintic curve. Let L_0, L_1 be two distinct lines which intersect C transversely and satisfy that $C \cap L_0 \cap L_1 = \emptyset$. We are interested in the bidouble cover S of \mathbb{P}^2 branched along $C + L_0 + L_1$.

Specifically, the surface S can be constructed in the following way. Take the double cover \bar{X}_0 of \mathbb{P}^2 branched along the sextic curve $C + L_1$. The surface \bar{X}_0 is a singular $K3$ surface with five A_1 singularities. Let X_0 be the $K3$ surface obtained by blowing up the singularities (i.e. take the canonical resolution of \bar{X}_0). Denote by E_1, \dots, E_5 the exceptional curves on X_0 with self intersection (-2) . Set D_0 to be the preimage of L_0 in X_0 and let $D_1 \subset X_0$ be the strict transform of L_1 . Choose $\mathcal{L} := \mathcal{O}_{X_0}(D_1 + E_1 + \dots + E_5)$. By computing the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ one sees that that $D_0 \sim 2D_1 + E_1 + \dots + E_5$ and hence $\mathcal{L}^{\otimes 2} = \mathcal{O}_{X_0}(D_0 + E_1 + \dots + E_5)$. Now we take the branched double cover S_0 of X_0 along $D_0 + E_1 + \dots + E_5$. The exceptional curves E_1, \dots, E_5 become (-1) -curves on S_0 . Contract these (-1) -curves and one obtains a surface S . For later use let us denote by σ_0 the involution of S so that $\bar{X}_0 = S/\sigma_0$. To summarize we have the following diagram (the left one).

$$(1.1) \quad \begin{array}{ccc} S_0 & \xrightarrow{f_0} & S \curvearrowright \sigma_0 \\ \downarrow \varphi_0 & & \downarrow \psi_0 \\ X_0 & \xrightarrow{g_0} & \bar{X}_0 \\ \downarrow \tau_0 & & \downarrow \delta_0 \\ \tilde{\mathbb{P}}^2 & \xrightarrow{h_0} & \mathbb{P}^2 \end{array} \quad \begin{array}{ccc} \sigma_1 \curvearrowright S & \xleftarrow{f_1} & S_1 \\ \downarrow \psi_1 & & \downarrow \varphi_1 \\ \bar{X}_1 & \xleftarrow{g_1} & X_1 \\ \downarrow \delta_1 & & \downarrow \tau_1 \\ \mathbb{P}^2 & \xleftarrow{h_1} & \tilde{\mathbb{P}}^2 \end{array}$$

Lemma 1.2. *The surface $\psi_0 \circ \delta_0 : S \rightarrow \mathbb{P}^2$ is a Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. By a result of Zariski the fundamental group $\pi_1(\mathbb{P}^2 - (C + L_0 + L_1))$ is abelian (see for example [Ful80]). Therefore, the covering map $\psi_0 \circ \delta_0$ is defined by

a normal subgroup and is Galois. Clearly, the Galois group is an abelian group of order 4. But it can not be $\mathbb{Z}/4\mathbb{Z}$ because otherwise the branched loci $C + L_0 + L_1$ is divisible by 4 in $\text{Pic}(\mathbb{P}^2)$ (or one can directly check that $\sigma_0^2 = \text{id}$ in the group of deck transformations $\text{Deck}(S/\mathbb{P}^2)$). \square

Since the Galois group of the bidouble cover S is $(\mathbb{Z}/2\mathbb{Z})^2$, there is a symmetric construction for S . Namely, one takes the double cover $\delta_1 : \bar{X}_1 \rightarrow \mathbb{P}^2$ branched along $C + L_0$ and resolves the five A_1 singularities to obtain a $K3$ surface $g_1 : X_1 \rightarrow \bar{X}_1$. Call the exceptional curves $F_1, \dots, F_5 \subset X_1$. It can be shown that $(\delta_1 \circ g_1)^{-1}(L_1) + F_1 + \dots + F_5$ is divisible by 2 in $\text{Pic}(X_1)$. Let S_1 be the double cover of X_1 along $(\delta_1 \circ g_1)^{-1}(L_1) + F_1 + \dots + F_5$. The surface S is obtained by contracting the (-1) -curves on S_1 . Let us use σ_1 to denote the involution of S with $\bar{X}_1 = S/\sigma_1$. (Note that σ_0 and σ_1 generate the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$.) See the right part of Diagram (1.1).

Proposition 1.3. *Let C be a smooth quintic curve and L_0, L_1 be two transverse lines with $C \cap L_0 \cap L_1 = \emptyset$. Let S be the $(\mathbb{Z}/2\mathbb{Z})^2$ -cover of \mathbb{P}^2 branched along $C + L_0 + L_1$. Then the surface S is a minimal algebraic surface of general type with $p_g(S) = 2$ and $K_S^2 = 1$. Moreover, S is simply connected and the canonical bundle K_S is ample.*

Proof. Notations as above. Denote by $\pi : S \rightarrow \mathbb{P}^2$ the covering map $\delta_0 \circ \psi_0 = \delta_1 \circ \psi_1$. It is clear from the construction that S is smooth. By [Mor88, Lem. 3.2] the canonical bundle K_S of the double cover S can be computed as $2K_S \sim \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$. It follows that K_S is ample and $K_S^2 = 1$. In particular, K_S is big and nef and hence S is a minimal surface of general type. Now let us compute $h^{2,0}(S)$ which clearly equals $h^0(S_0, \mathcal{O}_{S_0}(K_{S_0}))$. Let $\mathcal{L} = \mathcal{O}_{X_0}(D_1 + E_1 + \dots + E_5)$ be the line bundle associated with the double cover $\varphi_0 : S_0 \rightarrow X_0$ of the $K3$ surface X_0 . Note that $H^0(S_0, \mathcal{O}_{S_0}(K_{S_0})) = H^0(X_0, \varphi_{0*} \mathcal{O}_{S_0}(K_{S_0})) = H^0(X_0, \varphi_{0*} \varphi_0^*(\mathcal{L}))$. Since $\varphi_{0*} \varphi_0^*(\mathcal{L}) = \mathcal{L} \otimes \varphi_{0*}(\mathcal{O}_{X_0}) = \mathcal{L} \otimes (\mathcal{O}_{X_0} \oplus \mathcal{L}^{-1}) = \mathcal{L} \oplus \mathcal{O}_{X_0}$, we have $h^0(S_0, \mathcal{O}_{S_0}(K_{S_0})) = h^0(X_0, \mathcal{O}_{X_0}) + h^0(X_0, \mathcal{L})$. Because $D_1 + E_1 + \dots + E_5$ is effective and $(D_1 + E_1 + \dots + E_5)^2 = -2$, the space $H^0(X_0, \mathcal{L})$ is 1-dimensional. Thus $h^{2,0}(S) = h^0(S_0, K_{S_0}) = 2$ (this is also mentioned in [Cat79, Rmk. 8]). By [Bom73, Thm. 11, Thm. 14] the surface S is simply connected. \square

Algebraic surfaces of general type with $p_g = 2$ and $K^2 = 1$ have been studied by Horikawa [Hor76]. We call the bidouble covers S constructed above *special Horikawa surfaces*.

Proposition 1.4. *The canonical model of an algebraic surface Y of general type with $q(Y) = 0$, $p_g(Y) = 2$ and $K_Y^2 = 1$ is a hypersurface of degree 10 in $\mathbb{P}(1, 1, 2, 5)$. If K_Y is ample, then Y is isomorphic to a quasi-smooth hypersurface of degree 10 in $\mathbb{P}(1, 1, 2, 5)$.*

Proof. This has been proved in [Hor76, §2]. See also [BHPVdV04, §VII.7]. \square

Remark 1.5. A weighted hypersurface $Y \subset \mathbb{P}$ is *quasi-smooth* if the associated affine quasicone is smooth outside the vertex 0 (cf. [Dol82]). If in addition $\text{codim}_Y(Y \cap \mathbb{P}_{\text{sing}}) \geq 2$, then $Y_{\text{sing}} = Y \cap \mathbb{P}_{\text{sing}}$. (In our case, we have $\mathbb{P} = \mathbb{P}(1, 1, 2, 5)$ which has two singular points $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$ and hence $S_{\text{sing}} = S \cap \mathbb{P}_{\text{sing}}$.) Moreover, the cohomology $H^k(Y, \mathbb{C})$ of a quasi-smooth hypersurface Y admits a pure Hodge structure and explicit calculation can be done using (a slightly generalized version of) Griffiths residue.

We have shown that special Horikawa surfaces S have ample canonical bundles. A natural question is which degree 10 quasi-smooth hypersurfaces in $\mathbb{P}(1, 1, 2, 5)$ do they correspond to.

Proposition 1.6. *Let S be a $(\mathbb{Z}/2\mathbb{Z})^2$ -cover of \mathbb{P}^2 branched over a smooth quintic C and two general lines L_0, L_1 (i.e. S is a special Horikawa surface). Then S is isomorphic to a quasi-smooth hypersurface $z^2 = F(x_0^2, x_1^2, y)$ in $\mathbb{P}(1, 1, 2, 5) = \text{Proj}(\mathbb{C}[x_0, x_1, y, z])$ where F is a quintic polynomial.*

Proof. Denote by $\pi : S \rightarrow \mathbb{P}^2$ the covering map $\delta_0 \circ \psi_0 = \delta_1 \circ \psi_1$. For $i = 0, 1$ we let Λ_i be the reduced inverse image $\pi^{-1}(L_i)$ of L_i in S . By the proof of Proposition 1.3 we have $\Lambda_i \in |K_S|$. Choose a section $x_i \in H^0(S, \mathcal{O}_S(K_S))$ which cuts out Λ_i . Clearly $\{x_0, x_1\}$ forms a basis of $H^0(S, \mathcal{O}_S(K_S))$. Since $2K_S \sim \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, the covering map π is defined by a subspace of $|2K_S|$. Choose $y \in H^0(S, \mathcal{O}_S(2K_S))$ so that the subspace of $|2K_S|$ is generated by x_0^2, x_1^2 and y . Note that $y \neq x_0x_1$. By choosing a suitable $z \in H^0(S, \mathcal{O}_S(5K_S))$ we assume that the equation for S is $z^2 = F'(x_0, x_1, y)$ where F' is a weighted homogeneous polynomial of degree 10 in $\mathbb{P}(1, 1, 2, 5)$. (The defining equation must contain z^2 otherwise S is not quasi-smooth. Then we complete the square for z which does not affect the other coordinates.) The ramification locus π consists of three components $(x_0 = 0)$, $(x_1 = 0)$ and $(z = 0)$ which are mapped to L_0, L_1 and C respectively. The proposition then follows.

Alternatively, one considers the action of σ_i (see Diagram (1.1)) on the canonical ring $\bigoplus_{m \geq 0} H^0(S, \mathcal{O}_S(mK_S))$, hence on $\mathbb{P}(1, 1, 2, 5)$. Note that the involution σ_0 fixes Λ_0 pointwise and $\sigma_0(\Lambda_1) = \Lambda_1$. (Similarly for the involution σ_1 .) As in the proof of [Cat79, Thm. 3] one can choose y, z such that σ_0 acts on $\mathbb{P}(1, 1, 2, 5)$ by $[x_0, x_1, y, z] \mapsto [-x_0, x_1, y, z]$. Since σ_0 acts on S we assume that the defining equation for S is an eigenvector for σ_0 . It is not difficult to see that only even powers of x_0 can appear in the defining equation. Next one normalizes the equation and gets $z^2 = F'(x_0, x_1, y)$. Since F' is a weighted homogeneous polynomial of degree 10, x_1 also has even powers. \square

Remark 1.7.

- (1) Because S is quasi-smooth, the quintic F must contain y^5 (more generally, see [FPR15, Thm. 3.3]).
- (2) We have shown that a special Horikawa surface S is isomorphic to a quasi-smooth hypersurface in $\mathbb{P}(1, 1, 2, 5)$ with the equation $z^2 = F(x_0^2, x_1^2, y)$. The covering map $S \rightarrow \mathbb{P}^2$ is given by $[x_0, x_1, y, z] \mapsto [x_0^2, x_1^2, y]$. The Galois group is generated by $\sigma_{x_0} : [x_0, x_1, y, z] \mapsto [-x_0, x_1, y, z]$ and $\sigma_{x_1} : [x_0, x_1, y, z] \mapsto [x_0, -x_1, y, z]$ which correspond to σ_0 and σ_1 respectively.
- (3) Special Horikawa surfaces have moduli dimension $((\binom{2+5}{2} - 1) + 2 + 2 - 8 = 16$ (dimension of moduli for plane quintics together with two lines minus dimension of $\text{PGL}(3)$). The dimension of the moduli for degree 10 quasi-smooth hypersurfaces in $\mathbb{P}(1, 1, 2, 5) = \text{Proj}(\mathbb{C}[x_0, x_1, y, z])$ cut out by $z^2 = F(x_0^2, x_1^2, y)$ is $(\binom{2+5}{2} - 5 = 16$ (dimension of the quintic polynomials F minus dimension of the subgroup of the automorphism group of $\mathbb{P}(1, 1, 2, 5)$ acting on $z^2 = F(x_0^2, x_1^2, y)$: a semidirect product of the group consisting of the elements $[x_0, x_1, y, z] \mapsto [ax_0, bx_1, cy + dx_0^2 + ex_1^2, z]$ with the group generated by $[x_0, x_1, y, z] \mapsto [x_1, x_0, y, z]$).

Now let us study how the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ acts on the Hodge structures of special Horikawa surfaces S . We view S as a degree 10 quasi-smooth hypersurface in $\mathbb{P}(1, 1, 2, 5)$ cut out by the equation $z^2 = F(x_0^2, x_1^2, y)$. Choose the Kähler form corresponding to the canonical curve $(x_0 = 0)$ or $(x_1 = 0)$ (which are the reduced inverse image of L_0 and L_1 respectively) and the primitive cohomology can be described using Griffiths residue.

Proposition 1.8. *Let Y be a quasi-smooth hypersurface of degree d in a weighted projective space $\mathbb{P}(a_0, a_1, \dots, a_n)$. That is, Y is given by a weighted homogeneous polynomial $G(z_0, z_1, \dots, z_n)$ of degree d whose partial derivatives have no common zero other than the origin. Let*

$$E = \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$$

be the Euler vector field. Let $dV = dz_0 \wedge \dots \wedge dz_n$ be the Euclidean volume form, and let $\Omega = i(E)dV$ (where i denotes interior multiplication) be the projective volume form (which has degree $a_0 + \dots + a_n$). Consider expressions of the form

$$\Omega(A) = \frac{A \cdot \Omega}{G^q}$$

where A is a homogeneous polynomial whose degree is such that $\Omega(A)$ is homogeneous of degree 0. Then the Poincaré residues of $\Omega(A)$ span $F^{n-q}H_{\text{prim}}^{n-1}(Y, \mathbb{C})$ where F^\bullet denotes the Hodge filtration. Moreover, the residue lies in F^{n-q+1} if and only if A lies in the Jacobian ideal J_G of G (the ideal generated by the first partial derivatives of G).

Proof. This is [ACT11, Prop. 1.2]. See also [Dol82]. \square

Proposition 1.9. *Let $S \in \mathbb{P}(1, 1, 2, 5)$ be a quasi-smooth hypersurface of degree 10 given by $z^2 = F(x_0^2, x_1^2, y)$. Set σ_0 and σ_1 to be the automorphisms defined by $[x_0, x_1, y, z] \mapsto [-x_0, x_1, y, z]$ and $[x_0, x_1, y, z] \mapsto [x_0, -x_1, y, z]$ respectively. Let χ_0 and χ_1 be the corresponding characters of the Galois group defined by $\chi_0(\sigma_0) = 1$, $\chi_0(\sigma_1) = -1$ and similarly for χ_1 . Then we have the following decomposition of Hodge structures ($H_{\chi_l}^2(S, \mathbb{Q})$ is the eigenspace corresponding to χ_l for $l = 0, 1$)*

$$H_{\text{prim}}^2(S, \mathbb{Q}) = H_{\chi_0}^2(S, \mathbb{Q}) \oplus H_{\chi_1}^2(S, \mathbb{Q}).$$

Moreover, $H_{\chi_l}^2(S, \mathbb{Q})$ ($l = 0, 1$) has Hodge numbers $[1, 14, 1]$.

Proof. Notations as in Proposition 1.8. The decomposition is obtained by a Griffiths residue calculus. Specifically, let $G = F(x_0^2, x_1^2, y) - z^2$. Take a basis for $H_{\text{prim}}^2(S, \mathbb{C})$ of the forms $\text{Res} \frac{A \cdot \Omega}{G^q}$ with A being certain monomials $x_0^i x_1^j y^k$. The cohomology group $H^{2,0}(S)$ (resp. $H_{\text{prim}}^{1,1}(S), H^{0,2}(S)$) correspond to $q = 1$ (resp. $q = 2, q = 3$) and hence $i + j + 2k = 1$ (resp. $i + j + 2k = 11, i + j + 2k = 21$). In particular, $i + j$ must be odd. It follows that only the characters χ_0 and χ_1 appear in the decomposition of $H_{\text{prim}}^2(S, \mathbb{C})$ (and hence $H_{\text{prim}}^2(S, \mathbb{Q})$). The eigenspace $H_{\chi_0}^2(S, \mathbb{Q})$ is a sub-Hodge structure because $H_{\chi_0}^2(S, \mathbb{Q}) = \ker(\sigma_0^* - \text{id}) (= \ker(\sigma_1^* + \text{id}))$. Similarly for $H_{\chi_1}^2(S, \mathbb{Q})$. The claim on Hodge numbers can be checked for a special surface $z^2 = x_0^{10} + x_1^{10} + y^5$ (consider the induced action of σ_0 or σ_1 on the polarized variation of rational Hodge structure with fibers $H_{\text{prim}}^2(S, \mathbb{Q})$). \square

Remark 1.10. Let X_0 and X_1 be the $K3$ surfaces associated to a special Horikawa surface S in Diagram (1.1). One can show that $T_{\mathbb{Q}}(S) \cong T_{\mathbb{Q}}(X_0) \oplus T_{\mathbb{Q}}(X_1)$ where $T_{\mathbb{Q}} = \text{Tr} \otimes \mathbb{Q}$ denotes the rational transcendental lattice. Specifically, we consider the induced action σ_0^* and σ_1^* on $T_{\mathbb{Q}}(S)$. The space $T_{\mathbb{Q}}(S)$ decomposes as a direct sum of the eigenspaces of σ_0^* with eigenvalues 1 and -1 . By [Shi86, Prop. 5] we have $T_{\mathbb{Q}}(X_0) \cong T_{\mathbb{Q}}(S)^{\sigma_0^*}$ and $T_{\mathbb{Q}}(X_1) \cong T_{\mathbb{Q}}(S)^{\sigma_1^*}$ (where $T_{\mathbb{Q}}(S)^{\sigma_i^*}$ denotes the invariant part). Proposition 1.9 allows us to identify $T_{\mathbb{Q}}(S)^{\sigma_1^*}$ with the (-1) -eigenspace of σ_0^* which completes the proof. Generically, $T_{\mathbb{Q}}(S)$ has Hodge numbers $[2, 28, 2]$ (one applies [Moo93, Prop. 4] to $z^2 = x_0^{10} + x_1^{10} + y^5$ to see the generic Picard number is 1) and $T_{\mathbb{Q}}(X_l)$ has Hodge numbers $[1, 14, 1]$ (cf. [Laz09, Cor. 4.15]) for $l = 0, 1$.

2. THE INFINITESIMAL TORELLI THEOREM

We shall show in this section that (unlike for special Kunev surfaces [Kyn77] [Cat79] [Tod80]) the infinitesimal Torelli holds for special Horikawa surfaces S .

Theorem 2.1. *Let S be a bidouble cover of \mathbb{P}^2 branched along a smooth quintic C and two transverse lines L_0 and L_1 with $C \cap L_0 \cap L_1 = \emptyset$. The natural map*

$$(2.2) \quad p : H^1(S, \mathcal{T}_S) \rightarrow \text{Hom}(H^0(S, \omega_S), H^1(S, \Omega_S^1)),$$

given by cup product, is injective.

Proof. Usui [Usu78] has proved the infinitesimal Torelli theorem for the periods of holomorphic d -forms on certain d -dimensional complete intersections ($d \geq 2$) in certain weighted projective spaces. One can check that the conditions in op. cit. Theorem 2.1 are satisfied for the special Horikawa surface S and then apply the theorem. Specifically, by Proposition 1.6 the surface $S \subset \mathbb{P}(1, 1, 2, 5)$ is defined by $z^2 = F(x_0^2, x_1^2, y)$ and it is not difficult to verify that $z^2 - F(x_0^2, x_1^2, y), x_0, x_1, y$ forms a regular sequence for $\mathbb{C}[x_0, x_1, y, z]$. (One can apply op. cit. Proposition 3.1 to check the conditions and prove the infinitesimal Torelli for the periods of holomorphic 2-forms on any smooth hypersurface with ample canonical bundle in $\mathbb{P}(1, 1, 2, 5)$.) \square

Pardini [Par91b, Thm 3.1] [Par98, Thm 4.2] has considered the infinitesimal Torelli problem for certain abelian covers (including bidouble covers). The conditions of the theorems do not hold for special Horikawa surfaces S but the strategy still works. (In particular, we need the notation of prolongation bundle discussed in [Par98, §2].) Let us sketch the proof which might be useful for finding more boundary cases of Pardini's theorems.

Proof. The idea is decompose the infinitesimal Torelli map

$$p : H^1(S, \mathcal{T}_S) \rightarrow \text{Hom}(H^0(S, \omega_S), H^1(S, \Omega_S^1))$$

using the Galois group action. The first step is to figure out the building data (see [Par91a, Def. 2.1]) of the bidouble cover $\pi : S \rightarrow \mathbb{P}^2$. The (reduced) branched locus of π consists of three irreducible components:

$$D_0 := L_0, \quad D_1 := L_1, \quad \text{and} \quad D_z := C.$$

Let σ_0, σ_1 be the involutions in Diagram (1.1). Let $\sigma_z := \sigma_0 \circ \sigma_1$. The Galois group G of the abelian cover $S \rightarrow \mathbb{P}^2$ consists of $\text{id}, \sigma_0, \sigma_1, \sigma_z$. Let χ_0, χ_1, χ_z be the

corresponding nontrivial characters of G (and we shall denote the character group by G^*). Write

$$\pi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{L}_{\chi_0}^{-1} \oplus \mathcal{L}_{\chi_1}^{-1} \oplus \mathcal{L}_{\chi_z}^{-1}$$

where \mathcal{L}_{χ}^{-1} denotes the eigensheaf on which G acts via the character χ . One easily verify that $\mathcal{L}_{\chi_0} = \mathcal{L}_{\chi_1} = \mathcal{O}_{\mathbb{P}^2}(3)$ and $\mathcal{L}_{\chi_z} = \mathcal{O}_{\mathbb{P}^2}(1)$.

Next we compute the direct images of various sheaves. Let $D = D_0 + D_1 + D_z$. For the trivial character $\chi = 1$, define $\Delta_1 = D$. Set

$$\Delta_{\chi_0} := D_0, \Delta_{\chi_1} := D_1, \Delta_{\chi_z} := D_z.$$

We also let

$$D_{1,1^{-1}} := \emptyset, D_{\chi_0, \chi_0^{-1}} := D_1 + D_z, D_{\chi_1, \chi_1^{-1}} := D_0 + D_z, D_{\chi_z, \chi_z^{-1}} := D_0 + D_1.$$

(For every pairs of characters $\chi, \phi \in G^*$, $D_{\chi, \phi}$ is defined in [Par91b] and [Par98]. The fundamental relations of the building data are $\mathcal{L}_{\chi} + \mathcal{L}_{\phi} \equiv \mathcal{L}_{\chi\phi} + D_{\chi, \phi}$, see [Par91a, Thm. 2.1].) For any character χ , we have (see [Par91a, Prop. 4.1])

- $(\pi_* \mathcal{T}_S)^{(\chi)} = \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_{\chi}) \otimes \mathcal{L}_{\chi}^{-1}$ (In particular, $(\pi_* \mathcal{T}_S)^{(\text{inv})} = \mathcal{T}_{\mathbb{P}^2}(-\log D)$);
- $(\pi_* \Omega_S^1)^{(\chi)} = \Omega_{\mathbb{P}^2}^1(\log D_{\chi, \chi^{-1}}) \otimes \mathcal{L}_{\chi}^{-1}$ (In particular, $(\pi_* \Omega_S^1)^{(\text{inv})} = \Omega_{\mathbb{P}^2}^1$);
- $(\pi_* \omega_S)^{(\chi)} = \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\chi^{-1}}$ (In particular, $(\pi_* \omega_S)^{(\text{inv})} = \omega_{\mathbb{P}^2}$).

Since the map π is finite, for every coherent sheaf \mathcal{F} on S one has $H^k(S, \mathcal{F}) = H^k(\mathbb{P}^2, \pi_* \mathcal{F})$ ($k = 0, 1, 2$). In particular, we have

$$H^1(S, \mathcal{T}_S) = H^1(\mathbb{P}^2, \pi_* \mathcal{T}_S), H^1(S, \Omega_S) = H^1(\mathbb{P}^2, \pi_* \Omega_S), H^0(S, \omega_S) = H^0(\mathbb{P}^2, \pi_* \omega_S).$$

Combining with the splittings of $\pi_* \mathcal{T}_S$, $\pi_* \Omega_S$ and $\pi_* \omega_S$, we obtain the following decompositions:

$$(2.3) \quad H^1(S, \mathcal{T}_S) = H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log D)) \oplus \left(\bigoplus_{\chi \in G^* \setminus \{1\}} H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_{\chi}) \otimes \mathcal{L}_{\chi}^{-1}) \right)$$

$$(2.4) \quad H^1(S, \Omega_S^1) = H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1) \oplus \left(\bigoplus_{\chi \in G^* \setminus \{1\}} H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi, \chi^{-1}}) \otimes \mathcal{L}_{\chi}^{-1}) \right)$$

$$(2.5) \quad H^0(S, \omega_S) = H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}) \oplus \left(\bigoplus_{\chi \in G^* \setminus \{1\}} H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\chi^{-1}}) \right)$$

Since the cup product (and hence the infinitesimal Torelli map $p : H^1(S, \mathcal{T}_S) \rightarrow \text{Hom}(H^0(S, \omega_S), H^1(S, \Omega_S^1))$) is compatible with the group action, for characters $\chi, \phi \in G^*$ we consider

$$(2.6) \quad \begin{aligned} p_{\chi, \phi} : H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_{\chi}) \otimes \mathcal{L}_{\chi}^{-1}) \\ \rightarrow \text{Hom}(H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}}), H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes \mathcal{L}_{\chi\phi}^{-1})) \end{aligned}$$

Clearly, one has

$$(2.7) \quad p = \bigoplus_{\chi, \phi \in G^*} p_{\chi, \phi}.$$

Lemma 2.8. *The infinitesimal Torelli holds for S (i.e. the map p is injective) if and only if $\bigcap_{\phi \in G^*} \ker p_{\chi, \phi} = \{0\}$ for any character χ .*

Let us take a closer look at the maps $p_{\chi,\phi}$. For every pairs of characters $\chi, \phi \in G^*$ and every section $\xi \in H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}})$, consider the following diagram

$$\begin{array}{ccc} \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_{\chi}) \otimes \mathcal{L}_{\chi}^{-1} & \longrightarrow & \Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes \omega_{\mathbb{P}^2}^{-1} \otimes (\mathcal{L}_{\chi\phi} \otimes \mathcal{L}_{\phi^{-1}})^{-1} \\ \downarrow & & \downarrow \\ \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_{\chi}) \otimes \mathcal{L}_{\chi}^{-1} \otimes \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}} & \longrightarrow & \Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes \mathcal{L}_{\chi\phi}^{-1} \end{array}$$

where the vertical maps are given by multiplication by ξ and the horizontal maps are defined by contraction of tensors and by the fundamental relations [Par91a, Thm. 2.1] of the building data (N.B. there is a conical isomorphism between $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\chi\phi, (\chi\phi)^{-1}})$ and $\Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes (\omega_{\mathbb{P}^2}(D_{\chi\phi, (\chi\phi)^{-1}}))^{-1}$). Consider

$$(2.9) \quad \begin{aligned} q_{\chi,\phi} : H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_{\chi}) \otimes \mathcal{L}_{\chi}^{-1}) \\ \rightarrow H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes \omega_{\mathbb{P}^2}^{-1} \otimes (\mathcal{L}_{\chi\phi} \otimes \mathcal{L}_{\phi^{-1}})^{-1}) \end{aligned}$$

and

$$(2.10) \quad \begin{aligned} r_{\chi,\phi} : H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi, \chi^{-1}}) \otimes (\omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\chi} \otimes \mathcal{L}_{\phi^{-1}})^{-1}) \\ \rightarrow \text{Hom}(H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}}), H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi, \chi^{-1}}) \otimes \mathcal{L}_{\chi}^{-1})) \end{aligned}$$

Obviously we have $p_{\chi,\phi} = r_{\chi,\phi} \circ q_{\chi,\phi}$.

Let us analyze the maps $r_{\chi,\phi}$.

- The maps $r_{1,1}, r_{1,\chi_0}, r_{1,\chi_1}, r_{1,\chi_z}$ are injective: by explicit computation and Bott's vanishing theorem.
- The maps $r_{\chi_0,1}, r_{\chi_1,1}, r_{\chi_z,1}, r_{\chi_0,\chi_z}, r_{\chi_1,\chi_z}$ are zero maps: in these cases $H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}}) = 0$.
- The maps $r_{\chi_0,\chi_0}, r_{\chi_0,\chi_1}, r_{\chi_1,\chi_0}, r_{\chi_1,\chi_1}$ are injective: we use the prolongation bundle of the irreducible components of $D_{\chi,\chi^{-1}}$ defined in [Par98, §2] and show that if the multiplication map

$$\begin{aligned} H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}}) \otimes \left(\bigoplus_{\substack{B \text{ irreducible} \\ \text{components of } D_{\chi,\chi^{-1}}}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\chi}) \right) \\ \rightarrow \bigoplus_{\substack{B \text{ irreducible} \\ \text{components of } D_{\chi,\chi^{-1}}}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \omega_{\mathbb{P}^2}^{\otimes 2} \otimes \mathcal{L}_{\chi} \otimes \mathcal{L}_{\phi^{-1}}) \end{aligned}$$

is surjective, then the map $r_{\chi,\phi}$ is injective (cf. [Par98, §3]). As argued in [Par98, Prop. 3.5] the surjectivity of the multiplication map follows from a special case of [EL93, Thm. 2.1].

- The maps $r_{\chi_z,\chi_0}, r_{\chi_z,\chi_1}, r_{\chi_z,\chi_z}$ are injective: we also consider the prolongation bundle, and also note that $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \omega_{\mathbb{P}^2}^{\otimes 2} \otimes \mathcal{L}_{\chi} \otimes \mathcal{L}_{\phi^{-1}}) = 0$ (with B any irreducible component of $D_{\chi,\chi^{-1}}$) in these cases.

We use the method of [Par91b, §3] to study the maps $q_{\chi,\phi}$ (especially Diagram (3.4) and Lemma 3.1). The results are as follows.

- For $\chi = 1$, we have $\ker q_{1,\chi_0} \cap \ker q_{1,\chi_1} = \{0\}$ (and hence $\ker p_{1,\chi_0} \cap \ker p_{1,\chi_1} = \{0\}$).
- For $\chi = \chi_0$, both q_{χ_0,χ_1} and q_{χ_0,χ_z} are injective (and hence $p_{\chi_0,\chi_1} = r_{\chi_z,\chi_1} \circ q_{\chi_0,\chi_1}$ is injective).
- For $\chi = \chi_1$, both q_{χ_1,χ_0} and q_{χ_1,χ_z} are injective (and hence $p_{\chi_1,\chi_0} = r_{\chi_z,\chi_0} \circ q_{\chi_1,\chi_0}$ is injective).

- For $\chi = \chi_z$, both q_{χ_z, χ_0} and q_{χ_z, χ_1} are injective (and hence $p_{\chi_z, \chi_0} = r_{\chi_1, \chi_0} \circ q_{\chi_z, \chi_0}$ is injective, and $p_{\chi_z, \chi_1} = r_{\chi_0, \chi_1} \circ q_{\chi_z, \chi_1}$ is also injective).

The theorem clearly follows from these observations. \square

3. DEGREE 5 PAIRS AND A GENERIC GLOBAL TORELLI THEOREM

Let us review the period map for degree 5 pairs which will be used later in this section to prove a generic global Torelli problem for special Horikawa surfaces S .

Following [Laz09, Def. 2.1] we call a pair (C, L) consisting of a plane quintic curve C and a line $L \subset \mathbb{P}^2$ a *degree 5 pair*. Two such pairs are equivalent if they are projectively equivalent. We are interested in the degree 5 pairs (C, L) with $C + L$ defining a sextic curve admitting at worst *ADE* singularities. The coarse moduli space \mathcal{M}_{ADE} is contained in the GIT quotient $(\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5)) \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))) // \mathrm{SL}_3(\mathbb{C})$ (with respect to the linearization $\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(1)$).

To every degree 5 pair (C, L) such that $C + L$ has at worst *ADE* singularities, we associate a *K3* surface $X_{(C, L)}$ obtained by taking the canonical resolution of the double cover $\bar{X}_{(C, L)}$ of \mathbb{P}^2 along the sextic $C + L$. The period map for degree 5 pairs (C, L) is defined via the periods of $X_{(C, L)}$. Specifically, one first considers a generic pair (C, L) where C is smooth and L is a transversal. The *K3* surface $X_{(C, L)}$ contains 5 exceptional curves e_1, \dots, e_5 corresponding to the five intersection points $C \cap L$ and the strict transform l' of L . By [Laz09, Prop. 4.12, Def. 4.17], the lattice generated by $\{l', e_1, \dots, e_5\}$ is a primitive sublattice of $\mathrm{Pic}(X_{(C, L)})$ and hence $X_{(C, L)}$ is a lattice polarized *K3* surface (note that the lattice polarization depends on the labeling of the points of intersection $C \cap L$).

Notation 3.1. Let Λ be an even lattice. We define:

- Λ^* : the dual lattice;
- $A_\Lambda = \Lambda^*/\Lambda$: the discriminant group endowed with the induced quadratic form q_Λ ;
- $O(\Lambda)$: the group of isometries of Λ ;
- $O(q_\Lambda)$: the automorphisms of A_Λ that preserves the quadratic form q_Λ ;
- $O_-(\Lambda)$: the subgroup of isometries of Λ of spinor norm 1 (see also [Sca87, §3.6]);
- $\tilde{O}(\Lambda)$: the group of isometries of Λ that induce the identity on A_Λ ;
- $O^*(\Lambda) := O_-(\Lambda) \cap \tilde{O}(\Lambda)$.

We also introduce:

- Λ_{K3} : the *K3* lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$ (we denote the bilinear form by $(\cdot, \cdot)_{K3}$);
- M : the abstract lattice generated by $\{l', e_1, \dots, e_5\}$ which admits a unique primitive embedding into Λ_{K3} (one can also show that M is the generic Picard group of *K3* surfaces $X_{(C, L)}$ and $M \cong U(2) \oplus D_4$, see [Laz09, Cor. 4.15, Lem. 4.18]);
- $T = M_{\Lambda_{K3}}^\perp$: the orthogonal complement of M in Λ_{K3} (which is isomorphic to $U \oplus U(2) \oplus D_4 \oplus E_8$);
- $\mathcal{D}_M := \{\omega \in \mathbb{P}(T \otimes \mathbb{C}) \mid (\omega, \omega)_{K3} = 0, (\omega, \bar{\omega})_{K3} > 0\}$ which is the period domain for M -polarized *K3* surfaces;
- \mathcal{D}_M^0 : a connected component of \mathcal{D}_M which is a type IV Hermitian symmetric domain.

Let \mathcal{U} be an open subset of \mathcal{M}_{ADE} parameterizing the generic degree 5 pairs (C, L) with C smooth and with transverse intersections $C \cap L$. Let $\tilde{\mathcal{U}}$ be the \mathfrak{S}_5 -cover of \mathcal{U} that consists of triples (C, L, σ) with $\sigma : \{1, \dots, 5\} \rightarrow C \cap L$ labelings of $C \cap L$. By [Dol96] there is a period map $\tilde{\mathcal{U}} \rightarrow \mathcal{D}_M^0/O^*(T)$ sending (C, L, σ) to the periods of $X_{(C,L)}$ with the M -polarization determined by σ . By the global Torelli theorem and surjectivity of the period map for $K3$ surfaces and [Laz09, Prop. 4.14] the period map is birational. Note that there is a natural \mathfrak{S}_5 -action on $\tilde{\mathcal{U}}$. Moreover, the group $O^*(T)$ is a normal subgroup of $O_-(T)$ with $O_-(T)/O^*(T) \cong \mathfrak{S}_5$, and the residual \mathfrak{S}_5 -action on M is the permutation of the five points of intersection $C \cap L$ (op. cit. Proposition 4.22). (In fact, $O_-(T)$ is the monodromy group for the degree 5 pairs.) Thus, the period map is \mathfrak{S}_5 -equivariant and descends to a birational map $\mathcal{U} \rightarrow \mathcal{D}_M^0/O_-(T)$.

The birational map can be extended to a morphism $\mathcal{M}_{ADE} \rightarrow \mathcal{D}_M^0/O_-(T)$ using normalized M -polarizations. In particular, one needs to construct an M -polarization in the case of non-transversal intersections $C \cap L$. We briefly summarize the construction and refer the readers to [Laz09, §4.2.3] for the details. The construction is a modification of canonical resolution of singularities of double covers (see [BHPVdV04, Thm. III.7.2]). The role of modification is to keep track of the points of intersection $C \cap L$. More precisely, one chooses a labeling of the intersection $\sigma : \{1, 2, 3, 4, 5\} \rightarrow C \cap L$ such that for any $p \in C \cap L$ we have $|\sigma^{-1}(p)| = \text{mult}_p(C \cap L)$. Set $Y_0 = \mathbb{P}^2$ and $B_0 = C + L$. We blow up one singularity at a time (instead of doing simultaneous blow-ups) and do the first five blow-ups in points belonging to L . The new branched divisor B_i is the strict transform of B_{i-1} together with the exceptional divisor of the blow-up reduced mod 2. The process is repeated until the resulting divisor B_N is smooth. Denote the blow-up sequence by $Y_N \rightarrow \dots \rightarrow Y_i \rightarrow Y_{i-1} \rightarrow \dots \rightarrow Y_0 = \mathbb{P}^2$. The double cover $X_{(C,L)}$ of Y_N along B_N is a minimal resolution of $\bar{X}_{(C,L)}$. Let $p_i \in Y_{i-1}$ ($1 \leq i \leq 5$) be the centers of the blow-up which lies on the corresponding strict transform of L . Now we construct a primitive embedding of M into $\text{Pic}(X_{(C,L)})$ by sending l to the class of the reduced preimage of L and sending e_i ($1 \leq i \leq 5$) to the fundamental cycle associated to the simple singularity of B_{i-1} in the point p_i . The embedding is normalized in the sense of [Laz09, Def. 4.24] and the construction fits well in families.

By the global Torelli theorem for $K3$, the surface $X_{(C,L)}$ is unique up to isomorphism. Moreover, one can recover the degree 5 pair (C, L) because the classes $2l' + e_1 + \dots + e_5$ (which corresponds to the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and determines the covering map and the branched curve) and l' (which determines the line L and hence the residue quintic C) are fixed by the monodromy group $O_-(T)$. It follows that the period map

$$\mathcal{M}_{ADE} \hookrightarrow \mathcal{D}_M^0/O_-(T)$$

for degree 5 pairs (C, L) with $C + L$ admitting at worst ADE singularities is injective. This is the part we shall need later. For the completeness, let us mention that one can verify that the period map is surjective (see [Laz09, §4.3.1], especially Proposition 4.31). By Zariski's main theorem, the bijective birational morphism between two normal varieties $\mathcal{M}_{ADE} \rightarrow \mathcal{D}_M^0/O_-(T)$ is an isomorphism (op. cit. Theorem 4.1).

Let us focus on the generic global Torelli problem for special Horikawa surfaces. By [Par91a, Thm. 2.1] we construct the coarse moduli space \mathcal{M} for special Horikawa

surfaces as the open subset of the quotient¹

$$(\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5)) \times \text{Sym}^2 \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))) // \text{SL}_3(\mathbb{C})$$

corresponding to triples (C, L, L') which consist of smooth quintics C and transversals L and L' with $C \cap L \cap L' = \emptyset$. It is more convenient to work with a double cover \mathcal{M}' of \mathcal{M} . Specifically, \mathcal{M}' is the open subset of the GIT²

$$(\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5)) \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))) // \text{SL}_3(\mathbb{C})$$

which parameterizes (up to projective equivalence) triples (C, L_0, L_1) with C smooth quintics and L_0, L_1 “labelled” lines which intersect C transversely and satisfy $C \cap L_0 \cap L_1 = \emptyset$.

Choose a sufficiently general reference point $o \in \mathcal{M}'$ and let S_o be the corresponding bidouble cover. Let $V = H_{\text{prim}}^2(S_o, \mathbb{R})$ (with respect to the class of a canonical curve or equivalently a hyperplane section in $\mathbb{P}(1, 1, 2, 5)$) with polarization form Q . We also write $V_{\mathbb{Q}} = H_{\text{prim}}^2(S_o, \mathbb{Q})$. Consider the action of the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ on S_o and define $\rho : (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow \text{Aut}(V, Q)$ to be the corresponding representation. Notations as in Diagram (1.1). The Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ is generated by σ_0 and σ_1 . Denote the corresponding characters by χ_0 and χ_1 .

Notation 3.2. We shall use the following notations:

- $\mathcal{D} = \mathcal{D}(V, Q)$: the period domain parameterizing Q -polarized Hodge structures of weight 2 on V with hodge numbers $[2, 28, 2]$;
- $\mathcal{D}^\rho = \{x \in \mathcal{D} \mid \rho(a)(x) = x, \forall a \in (\mathbb{Z}/2\mathbb{Z})^2\}$;
- $V(\chi)$: the eigensubspace of V corresponding to the character χ (it is not difficult to see that the eigenspaces $V(\chi)$ and $V(\chi')$ are orthogonal with respect to Q if $\chi \neq \chi'$);
- $V(\chi)_{\mathbb{Q}} := V(\chi) \cap V_{\mathbb{Q}}$;
- $\mathcal{D}(\chi)$: the period domain $\mathcal{D}(V(\chi), Q|_{V(\chi)})$ of type $[1, 14, 1]$.

Lemma 3.3. *There is a natural map $\mathcal{D}^\rho \rightarrow \mathcal{D}(\chi_0) \times \mathcal{D}(\chi_1)$ which is injective.*

Proof. The lemma follows from Proposition 1.9 and [DK07, §7]. Specifically, only the characters χ_0 and χ_1 appear in the decomposition of the vector space V . Let $V \otimes \mathbb{C} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ be a Q -polarized Hodge structure on V . The map is defined by sending the Hodge structure to the induced Q -polarized Hodge structures on $V(\chi_0)$ and $V(\chi_1)$ which is clearly injective. \square

Now we show that the period spaces $\mathcal{D}(\chi_0)$ and $\mathcal{D}(\chi_1)$ are both isomorphic to the period space \mathcal{D}_M for M -polarized $K3$ surfaces.

Lemma 3.4. *There exists an isomorphism ($l = 0$ or 1)*

$$(V(\chi_l)_{\mathbb{Q}}, \frac{1}{2}Q) \cong (T \otimes \mathbb{Q}, (\cdot, \cdot)_{K3} \otimes \mathbb{Q}).$$

Proof. We take χ_0 as an example. Let $S = S_o$ be the bidouble cover corresponding to the reference point $o \in \mathcal{M}'$. Notations as in Diagram (1.1). In particular, by abuse of notation σ_0 also denotes the involution relative to $\varphi_0 : S_0 \rightarrow X_0$. Label the points of intersection $C \cap L_1$ (the isomorphism we shall describe does not depend on the labeling) and there is a primitive embedding of M (and hence

¹for the linearization induced by $\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_3^* \mathcal{O}_{\mathbb{P}^2}(1)$

²with respect to the linearization $\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_3^* \mathcal{O}_{\mathbb{P}^2}(1)$

T) into $H^2(X_0, \mathbb{Z})$. Since o is sufficient general, M is the Picard lattice of the $K3$ surface X_0 ([Laz09, Cor. 4.15]). Consider the composition of linear maps

$$T \otimes \mathbb{Q} \hookrightarrow H^2(X_0, \mathbb{Q}) \xrightarrow{\varphi_0^*} H^2(S_0, \mathbb{Q})^{\sigma_0^*}.$$

The map φ_0^* is an isomorphism of vector spaces. Let $D_1 \subset X_0$ be the strict transform of L_1 and set E_1, \dots, E_5 to be the exceptional curves. Clearly, $T \otimes \mathbb{Q}$ is the orthogonal complement of $\mathbb{Q}[D_1] \oplus \mathbb{Q}[E_1] \oplus \dots \oplus \mathbb{Q}[E_5]$ in $H^2(X_0, \mathbb{Q})$. Thus, $T \otimes \mathbb{Q}$ is mapped onto $V(\chi_0)_{\mathbb{Q}} = H_{\text{prim}}^2(S, \mathbb{Q})^{\sigma_0^*}$ by φ_0^* . The claim on the bilinear forms is clear. \square

Corollary 3.5. *The period domain $\mathcal{D}(\chi_l)$ is isomorphic to \mathcal{D}_M for $l = 0, 1$.*

To formulate the theorem we also need to choose a discrete group. Let Γ_0 (resp. Γ_1) be the discrete subgroup of $\text{Aut}(V(\chi_0)_{\mathbb{Q}}, Q)$ (resp. $\text{Aut}(V(\chi_1)_{\mathbb{Q}}, Q)$) corresponding to $O_-(T)$ (using Lemma 3.4). Set Γ to be the discrete subgroup in $\text{Aut}(V_{\mathbb{Q}}, Q)$ which projects onto Γ_0 and Γ_1 under the isomorphism $V_{\mathbb{Q}} = V(\chi_0)_{\mathbb{Q}} \oplus V(\chi_1)_{\mathbb{Q}}$. Now we consider the period map for special Horikawa surfaces which are canonically polarized. By the fact that the monodromy representation (for the very general base point) commutes with the representation ρ ([DK07, §7]) and [Laz09, Prop. 4.22], the discrete subgroup Γ contains the image of the monodromy representation.

Theorem 3.6. *The period map $\mathcal{P} : \mathcal{M}' \rightarrow \mathcal{D}^{\rho}/\Gamma$ is generically injective.*

Proof. Let us consider the map

$$\mathcal{P}_0 \times \mathcal{P}_1 : \mathcal{M}' \rightarrow \mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T)$$

which is defined using the period maps for the degree 5 pairs (C, L_0) and (C, L_1) . We have the following diagram (where the superscript 0 denotes a connected component)

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{\mathcal{P}} \mathcal{D}^{\rho,0}/\Gamma & \hookrightarrow \mathcal{D}^0(\chi_0)/\Gamma_0 \times \mathcal{D}^0(\chi_1)/\Gamma_1 \\ & \searrow \mathcal{P}_0 \times \mathcal{P}_1 & \downarrow \\ & & \mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T) \end{array}$$

By the proof of Lemma 3.4, the vertical arrow is defined by identifying the transcendental lattice T with the invariant parts of the underlying vector space $V_{\mathbb{Q}} = H_{\text{prim}}^2(S_0, \mathbb{Q})$ for the involutions σ_l^* ($l = 0, 1$) via the natural pull-backs. It follows that the diagram is commutative. By [Laz09] Section 4.2.3 or Theorem 4.1, one can recover a degree 5 pair (C, L) (up to projective equivalence) from the periods of the $K3$ surface $X_{(C,L)}$. As a result, the isomorphism class of a triple (C, L_0, L_1) is determined by the periods of the $K3$ surfaces $X_{(C,L_0)}$ and $X_{(C,L_1)}$ provided that the quintic C has no nontrivial automorphism. Thus, the map $\mathcal{P}_0 \times \mathcal{P}_1 : \mathcal{M}' \rightarrow \mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T)$ is generically injective and so is the period map \mathcal{P} . \square

Remark 3.7. Let \mathcal{W} be the subset of $|\mathcal{O}_{\mathbb{P}^2}(5)| \times |\mathcal{O}_{\mathbb{P}^2}(1)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|$ corresponding to triples (C, L_0, L_1) with $C + L_0$ and $C + L_1$ admitting at worst ADE singularities and $C \cap L_0 \cap L_1 = \emptyset$. By taking bidouble covers we obtain a family of surfaces $S \rightarrow \mathcal{W}$ with only du Val singularities. By applying a simultaneous resolution to the

family \mathcal{S} we obtain a family $\tilde{\mathcal{S}} \rightarrow \mathcal{W}$ (after a finite base change of \mathcal{W}) of Horikawa surfaces (which are surfaces of general type with $p_g = 2$ and $K^2 = 1$). Consider the period map $\mathcal{P}_0 \times \mathcal{P}_1 : \mathcal{W} \rightarrow \mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T)$. The generic global Torelli theorem holds for this family. Namely, if two generic points in \mathcal{W} have the same image in $\mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T)$ then the corresponding triples are projectively equivalent.

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